CHAPTER 3

Separation Axioms

BY

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Module-4: Urysohn's Lemma

Now we shall gradually move to Urysohn Metrization Theorem. In this course we shall require the famous Urysohn's Lemma. We have already given a version of Urysohn's lemma for metric space. But that depends completely on the metric. In this module we shall present a general version of Urysohn's lemma.

Theorem 1 (Urysohn's Lemma). *A space X is normal if and only if for any two disjoint closed sets E and F there exists a continuous functions $f : X :\to \mathbb{R}$ such that*

$$f(E) = 0 \text{ and } f(F) = 0.$$

This is a deep Theorem, both from the point of view that its proof, which involves really original idea, also from the point of view of its application.

Proof. Let X be a normal space. Set $V = X \setminus F$, an open set containing E. Then by normality criteria there exists an open set $U_{\frac{1}{2}}$ such that

$$E \subset U_{\frac{1}{2}} \subset \overline{U_{\frac{1}{2}}} \subset V.$$

By successive application of normality criteria gives that there exist open sets $U_{\frac{1}{4}}$ and $U_{\frac{3}{4}}$ such that

$$E \subset U_{\frac{1}{4}} \subset \overline{U}_{\frac{1}{4}} \subset U_{\frac{1}{2}} \subset \overline{U_{\frac{1}{2}}} \subset U_{\frac{3}{4}} \subset \overline{U}_{\frac{3}{4}} \subset V.$$

Continuing this manner for each dyadic rational number $r \in (0, 1)$, an open set U_r such that

$$U_r \subset U_s,\ 0 < r < s < 1,$$

$$E \subset U_r,\ 0 < r < 1,$$

$$U_r \subset V,\ 0 < r < 1.$$

With this information we shall now go to define the function f.

$$f(x) = \begin{cases} 0 & \text{if } x \in U_r \text{ for all } r > 0 \\ \sup\{r : x \notin U_r\} \end{cases}$$

Evidently $0 \leq f \leq 1$, $f = 0$ on E and $f = 1$ on F. It suffices to show that f is continuous.

Let $x \in X$. For convenience, we assume that $0 < f(x) < 1$, the case $f(x) = 0$ and $f(x) = 1$ are not difficult. Let $\varepsilon > 0$. Choose dyadic rational number $0 < r < s < 1$ and

$$f(x) - \varepsilon < r < f(x) < f(x) + \varepsilon.$$

Then $x \notin U_t$ for dyadic rational numbers between r and $f(x)$, so that $x \notin \overline{U}_r$. On the otherhand $x \in U_s$. Hence $W = U_s \setminus \overline{U}_r$ is an open neighborhood of x. If $y \in W$, then from the definition of f we see that $r \leq f(x) \leq s$. In particular, $|f(x) - f(y)| < \varepsilon$ for $y \in W$, so that f is continuous at x. □

Recall that A is a G_δ set in a space X if A is the intersection of a countable collection of open sets of X. In metric space all closed sets are G_δ sets. But this is not true in general. In normal space we have the following Theorem.

Theorem 2. *Let X be normal space. There exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) > 0$ for $x \notin A$, if and only if A is a closed G_δ set in X.*

Proof. Suppose there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) > 0$ for $x \notin A$. Then $A = f^{-1}(0)$ must be closed. Now $A = \bigcap_n f^{-1}(-\frac{1}{n}, \frac{1}{n})$ and each $f^{-1}(-\frac{1}{n}, \frac{1}{n})$ is an open set. Hence A is a G_δ set also.
Conversely let A be a closed G_δ set. Then there exists a sequence (U_n) of open sets such that $A = \bigcap_n U_n$. Then for each n there exists a continuous function f_n which vanishes

on A and equal to 1 on $X - U_n$. Now take

$$f = \sum_n \frac{|f_n|}{2^n}.$$

Then clearly f is continuous and serves our purpose. $\square$

Now we are in a position to prove the coveted Urysohn Metrization Theorem.

Theorem 3. *Every regular second countable space is metrizable.*

Proof. We know that any regular second countable space X is normal so that we can invoke Urysoh's lemma. Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable base. Then for any $m, n \in \mathbb{N}$ if $\overline{B_m} \subset B_n$ there exists some $f \in C^*(X)$ such that $f(\overline{B_m}) = 0$ and $f(B_n) = 1$. In this way we get a countable collention of functions say $\{f_n : n \in \mathbb{N}\}$ which does the above job. We define $e : X \to \mathbb{R}^{\mathbb{N}}$ to be $\pi_n(e(x)) = f_n(x)$. We shall prove that e is an embedding. Continuity follows from the fact that each factor map f_n is continuous. If $x \neq y$ then there exists $B_m B_n \in \mathcal{B}$ such that $x \in B_m$, $y \notin B_n$ and $\overline{B_m} \subset B_n$. So there exists some f_k such that $f_k(x) = 0$ and $f_k(y) = 1$.

It's remain to show that $e : X \to e(X)$ is open. Let U be open in X, then we need to show that set $e(U)$ is open in $e(X)$. Let $v \in e(U)$. We have to find an open set W of $e(X)$ such that $v \in W \subset e(U)$. Let u be the point of U such that $e(u) = v$. Choose an index N for which $f_N(u) > 0$ and $f_N(X - U) = \{0\}$. Take the open ray $(0, +\infty)$ in $\mathbb{R}$, and let V be the open set $\pi_N^{-1}((0, +\infty))$ of $\mathbb{R}^{\mathbb{N}}$. We claim that $v \in W \subset e(U)$ First, $v \in W$ because $\pi_N(u) = \pi_N(e(u)) = f_N(u) > 0$. Second, $W \subset e(U)$. For if $z \in W$, then $z = e(x)$ for some $x \in X$, and $\pi_N(z) \in (0, +\infty)$. Since $\pi_N(z) = \pi_N(e(x)) = f_N(x)$, and f_N vanishes outside U the point x must be in U. Then $z = e(x)$ is in $e(U)$ as desired.

Thus e is an imbedding of X in $\mathbb{R}^{\mathbb{N}}$. $\square$

Definition 1. *If E and F be two disjoint closed sets in a space X and there exists a continuous functions $f : X :\to \mathbb{R}$ such that $f(E) = 0$ and $f(F) = 0$. We say that E and F can be separated by a continuous function.*

The Urysohn lemma says that if every pair of disjoint closed sets in X can be separated by disjoint open sets, then each such pair can be separated by a continuous function. The converse is trivial, for if

$$f(E) = 0 \text{ and } f(F) = 0.$$

is the function, then $f^{-1}[0, \frac{1}{3})$ and and $f^{-1}(\frac{2}{3}, 0]$ are disjoint open sets containing A and B, respectively.

This fact leads to the analogue question for regular spaces, that is whether regularity is equivalent with the fact that points and closed sets can be separated by continuous functions. Unfortunately this does not hold in general, an example will be provided latter. This fact leads to the following definition.

Definition 2. *A space X is said to be Completely Regular if all finite sets are closed and for any $x \in X$ and a closed sets F not containing x, there exists a continuous functions $f : X :\rightarrow \mathbb{R}$ such that*

$$f(x) = 0 \text{ and } f(F) = 0.$$

This is quite clear that Completely regular spaces are regular and normal spaces are Completely Regular.

Theorem 4. *A subspace of a completely regular space is completely regular.*

Proof. Let X be a completely regular, let Y be a subspace of X. Let x be a point of Y, and let K be a closed set of Y disjoint from x. We choose a closed set in X such that $K = H \cap Y$. Therefore $x \notin H$. Since X is completely regular, we can choose a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f(H) = 0$. The restriction of f to Y is desired continuous function on Y. □

Theorem 5. *Product of completely regular spaces is completely regular.*

Proof. Let $X = \prod X_\alpha$ be a product of completely regular spaces. Let $x = (x_\alpha)$ be a point of X and let U be a open set of X containing x. Then $U = \prod U_\alpha$, where each U_α is open in X_α and $U_\alpha = X_\alpha$ except for finitely many α's, say, $\alpha_1, \alpha_2, \ldots, \alpha_k$. Then for each α_i we can choose continuous function $f_i : X \to \mathbb{R}$ such that $f_{\alpha_i}(x_{\alpha_i}) = 1$ and $f_{\alpha_i}(X_{\alpha_i} \setminus U_{\alpha_i}) = 0$. Now let us set $\phi_{\alpha_i} = f_{\alpha_i} \circ \pi_{\alpha_i}$. Then each ϕ_{α_i} is continuous from X to $\mathbb{R}$. If we set $\phi = \phi_{\alpha_1}.\phi_{\alpha_2}. \ldots .\phi_{\alpha_k}$, then it is easy to observe that $\phi(x) = 1$ and $\phi(X \setminus U) = 0$. □